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Independence of Yang-Mills Equations with Respect to the Invariant Pairing in the Lie Algebra

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It is proved that the Euler–Lagrange equations of a Yang-Mills type Lagragian is independent with respect to the chosen pairing in the Lie algebra. Moreover, the Hamilton-Cartan equations of these Lagrangians are obtained and proved to be also independent with respect to the pairing.

KEY WORDS: Adjoint-invariant pairing; gauge invariance; jet bundles; principal connection; Yang-Mills fields.

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1. INTRODUCTION AND PRELIMINARIES

Let (M, g) be a pseudo-Riemannian manifold of signature (n^+, n^-), $\dim M = n = n^+ + n^-$. Let $\mathbf{v}_g = \sqrt{|\det(g_{ij})|}dx^1 \wedge \ldots \wedge dx^n$, $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, be the pseudo-Riemannian volume form attached to g, and let $T_x M \to T_x^* M$, $X \mapsto X^\flat$ be the canonical isomorphism induced by g, with inverse map $T_x^* M \to T_x M$, $w \mapsto w^\sharp$. The metric g induces a new metric $g^{(r)}$ on $\wedge^r T^* M$ defined as follows:

$$g^{(r)}(w^1 \wedge \ldots \wedge w^r, \bar{w}^1 \wedge \ldots \wedge \bar{w}^r) = \det(g((w^i)^\sharp, (\bar{w}^j)^\sharp)_{i,j=1}^r).$$

The Hodge star associated to g can be extended to vector valued forms as follows: Let $V \to M$ be a vector bundle. We define $\star\colon \wedge^\bullet T^* M \otimes V \to \wedge^\bullet T^* M \otimes V$ as follows: $\star(\omega_r \otimes v) = (\star\omega_r) \otimes v$, $\forall \omega_r \in \wedge^\bullet T_x^* M$, $\forall v \in V_x$.

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Let $\pi : P \to M$ be a principal G-bundle and let $\pi_{\mathrm{ad}\,P} : \mathrm{ad}P \to M$ be the adjoint bundle; i.e., the bundle associated with P under the adjoint representation of G on its Lie algebra $\mathfrak{g}$. For every $B \in \mathfrak{g}$ and every $u \in P$, let $(u, B)_G$ be the coset of $(u, B) \in P \times \mathfrak{g}$ modulo G. A symmetric bilinear form $\langle \cdot, \cdot \rangle \in S^2\mathfrak{g}^*$ is said to be invariant under the adjoint representation if the following equation holds:

$$\langle \mathrm{Ad}_g B, \mathrm{Ad}_g C \rangle = \langle B, C \rangle, \quad \forall g \in G, \ \forall B, C \in \mathfrak{g}. \tag{1}$$

By taking derivatives, the equation (1) implies the following:

$$\langle [A, B], C \rangle + \langle B, [A, C] \rangle = 0, \quad \forall A, B, C \in \mathfrak{g}. \tag{2}$$

If the group G is connected, then the conditions (1) and (2) are equivalent.

Every symmetric bilinear form $\langle \cdot, \cdot \rangle \in S^2\mathfrak{g}^*$ invariant under the adjoint representation induces a fibred metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathrm{ad}P \oplus \mathrm{ad}P \to \mathbb{R}$ by setting

$$\langle\!\langle (u, B)_G, (u, C)_G \rangle\!\rangle = \langle B, C \rangle, \quad \forall u \in P, \ \forall B, C \in \mathfrak{g}. \tag{3}$$

Every pseudo-Riemannian metric g on M and every fibred $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathrm{ad}P$ induce a fibred metric on the vector bundle of $\mathrm{ad}P$-valued differential r-forms on M by setting $(\!(\alpha_r \otimes a, \beta_r \otimes b)\!) = g^{(r)}(\alpha_r, \beta_r)\langle\!\langle a, b \rangle\!\rangle$, for all $\alpha, \beta \in \wedge^r T_x^* M$ and all $a, b \in (\mathrm{ad}P)_x$. Moreover, the pairing 3 defines an exterior product $\dot{\wedge} : (\wedge^\bullet T^*M \otimes \mathrm{ad}P) \oplus (\wedge^\bullet T^*M \otimes \mathrm{ad}P) \to \wedge^\bullet T^*M$ by setting $\left(\alpha_q \otimes a\right) \dot{\wedge} (\beta_r \otimes b) = \left(\alpha_q \wedge \beta_r\right) \langle\!\langle a, b \rangle\!\rangle$; see Bleecker (1981).

Let $p : C \to M$ be the bundle of connections of P. According to the previous definitions, a pseudo-Riemannian metric g on M and an adjoint-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ allow one to define a quadratic Lagrangian density $L\mathbf{v}_g$ on J^1C by setting,

$$\begin{aligned}(L\mathbf{v}_g)(j_x^1 \sigma_\Gamma) &= (\!(\Omega^\Gamma(x), \Omega^\Gamma(x))\!)\mathbf{v}_g(x) \\ &= \Omega^\Gamma(x) \dot{\wedge} \star \Omega^\Gamma(x),\end{aligned} \tag{4}$$

where σ_Γ is a local section of p defining the (local) principal connection Γ, whose curvature form at $x \in M$ is denoted by $\Omega^\Gamma(x)$. We racall that the assignment $j_x^1 \sigma_\Gamma \mapsto \Omega^\Gamma(x)$ is in fact a fibred mapping $\Omega : J^1C \to \wedge^2 T^*M \otimes \mathrm{ad}P$ called the curvature mapping, cf. Bleecker (1981).

If $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathrm{CK}}$ is the Cartan-Killing pairing, then the previous Lagrangian is the standard Yang-Mills Lagrangian and if, in addition, G is semisimple, then the Euler-Lagrange equations are the well-known Yang-Mills equations. Theorem 1 below states that these equations can also be obtained when $\langle \cdot, \cdot \rangle_{\mathrm{CK}}$ is replaced by an arbitrary adjoint-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. The interest of this result is motivated from the fact that there are different invariant symmetric bilinear forms on $\mathfrak{g}$. Indeed, we can obtain the following classification scheme for invariant pairings in semisimple algebras:

- If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ is its decomposition into simple Lie algebras, then the adjoint-invariant symmetric bilinear forms on $\mathfrak{g}$ are as follows:

$$F((A_1, \ldots, A_k), (A'_1, \ldots, A'_k)) = \sum_{i=1}^{k} \langle A_i, A'_i \rangle_i, \quad A_i, A'_i \in \mathfrak{g}_i,$$

$\langle \cdot, \cdot \rangle_i$ being any adjoint-invariant symmetric bilinear form on $\mathfrak{g}_i$.

Concerning simple real Lie algebras, there are two cases only:

- $\mathfrak{g}$ is a real form of a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Then any adjoint-invariant bilinear form f on $\mathfrak{g}$ is a scalar multiple of the Cartan-Killing metric on $\mathfrak{g}$.
- $\mathfrak{g}$ is the underlying simple real Lie algebra of a given simple complex Lie algebra $\bar{\mathfrak{g}}$. By using the theory of real semisimple Lie algebra (e.g., see Onishchik (2004)), the following can be proved: If f is an adjoint-invariant $\mathbb{R}$-bilinear form on $\mathfrak{g}$, then there exist $\lambda, \mu \in \mathbb{R}$ such that $f = \lambda Re(\langle \cdot, \cdot \rangle_{\mathrm{CK}}^{\bar{\mathfrak{g}}}) + \mu Im(\langle \cdot, \cdot \rangle_{\mathrm{CK}}^{\bar{\mathfrak{g}}})$. Since $\langle \cdot, \cdot \rangle_{\mathrm{CK}}^{\mathfrak{g}} = 2Re(\langle \cdot, \cdot \rangle_{\mathrm{CK}}^{\bar{\mathfrak{g}}})$, we conclude the existence of a new adjoint-invariant $\mathbb{R}$-bilinear form on $\mathfrak{g}$ essentially different from the Cartan-Killing metric on $\mathfrak{g}$. Indeed, $\mathrm{Im}(\langle \cdot, \cdot \rangle_{\mathrm{CK}}^{\mathfrak{g}})$ is not generally a multiple of $\mathrm{Re}(\langle \cdot, \cdot \rangle_{\mathrm{CK}}^{\mathfrak{g}})$; for example, this is readily checked for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

2. INDEPENDENCE OF E-L EQUATIONS

Theorem 2.1. *Let $\pi: P \to M$ be a principal bundle on a pseudo-Riemannian compact oriented connected manifold (M, g) and let $p: C \to M$ be its bundle of connections. Let $L\mathbf{v}_g$ be the Lagrangian density defined in the formula* (4)*, where $(\!(\cdot, \cdot)\!)$ is the fibred metric induced on $\wedge^\bullet T^*M \otimes \mathrm{ad}P$ by g and a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle \in S^2\mathfrak{g}^*$, which is invariant under the adjoint representation. Then, the Euler-Lagrange equations for $L\mathbf{v}_g$ are independent of $\langle \cdot, \cdot \rangle$.*

Proof: Let Γ be an arbitrary principal connection on $\pi: P \to M$. As the adjoint bundle is an associated bundle to P, Γ induces a covariant derivative ∇^Γ on $\mathrm{ad}P$; e.g., see Kobayashi and Nomizu (1963). First of all, we prove that for every pair of sections ξ, η of $\mathrm{ad}P \to M$ the following formula holds: $d\langle \xi, \eta \rangle = \langle \nabla^\Gamma \xi, \eta \rangle + \langle \xi, \nabla^\Gamma \eta \rangle$. Locally, let $\{\widetilde{B}_\alpha\}$ be a basis of sections of $\mathrm{ad}P \to M$ defined by a basis $\{B_\alpha\}$ of $\mathfrak{g}$. If we put $\xi = \xi^\alpha \widetilde{B}_\alpha$, $\eta = \eta^\alpha \widetilde{B}_\alpha$, one has (see Castrillón López and Muñoz Masqué (2001, formula (5.2))),

$$\nabla^\Gamma \xi = \left(\frac{\partial \xi^\alpha}{\partial x^i} + c^\alpha_{\beta\gamma} \xi^\beta A^\gamma_i \right) dx^i \otimes \widetilde{B}_\alpha,$$

and similarly for $\nabla^\Gamma \eta$, where A_i^γ are the local coordinates of Γ. Hence,

$$\begin{aligned}\langle \nabla^\Gamma \xi, \eta \rangle + \langle \xi, \nabla^\Gamma \eta \rangle &= \left\langle \left(\frac{\partial \xi^\alpha}{\partial x^i} + c^\alpha_{\beta\gamma} \xi^\beta A_i^\gamma \right) dx^i \otimes \widetilde{B}_\alpha, \eta^\tau \widetilde{B}_\tau \right\rangle \\ &\quad + \left\langle \xi^\tau \widetilde{B}_\tau, \left(\frac{\partial \eta^\alpha}{\partial x^i} + c^\alpha_{\beta\gamma} \eta^\beta A_i^\gamma \right) dx^i \otimes \widetilde{B}_\alpha \right\rangle \\ &= \langle d\xi^\alpha \widetilde{B}_\alpha, \eta \rangle + \langle \xi, d\eta^\alpha \widetilde{B}_\alpha \rangle \\ &\quad + A_i^\gamma dx^i \{ \langle [\widetilde{B}_\gamma, \xi], \eta \rangle + \langle \eta, [\widetilde{B}_\gamma, \eta] \rangle \},\end{aligned}$$

which yields $d\langle \xi, \eta \rangle$, because of the invariance under the adjoint representation. For arbitrary ad$P$-valued forms α, β, we readily obtain

$$d(\alpha \dot{\wedge} \beta) = (\nabla^\Gamma \alpha) \dot{\wedge} \beta + (-1)^{\deg(\alpha)} \alpha \dot{\wedge} (\nabla^\Gamma \beta). \tag{5}$$

The Lagrangian density $\Lambda = L\mathbf{v}_g$ can be written as $\Lambda(j_x^1 \sigma_\Gamma) = (\Omega^\Gamma \dot{\wedge} \star \Omega^\Gamma)_x$. For a variation $\Gamma \mapsto \Gamma + t\omega$, where ω is an ad$P$-valued 1-form on M, the action principle reads

$$0 = \left. \frac{d}{dt} \right|_{t=0} \int_M L\big(j_x^1 \sigma_{\Gamma + t\omega}\big) = \left. \frac{d}{dt} \right|_{t=0} \int_M \Omega^{\Gamma + t\omega} \dot{\wedge} \star \Omega^{\Gamma + t\omega} = 2 \int_M \nabla^\Gamma \omega \dot{\wedge} \star \Omega^\Gamma.$$

Making use of (5), we have

$$0 = \int_M d(\omega \dot{\wedge} \star \Omega^\Gamma) - \int_M \omega \dot{\wedge} \nabla^\Gamma \star \Omega^\Gamma = - \int_M \omega \dot{\wedge} \nabla^\Gamma \star \Omega^\Gamma,$$

which gives the Euler-Lagrange equation $\nabla^\Gamma \star \Omega^\Gamma = 0$, for ω is arbitrary and $\langle \cdot, \cdot \rangle$ is non-degenerate. □

3. INDEPENDENCE OF H-C EQUATIONS

Let $p\colon E \to M$ be a fibred manifold, $\dim M = n$, $\dim E = m + n$, where M is assumed to be connected and oriented by a volume form $\mathbf{v}$. The solutions to the Hamilton-Cartan equations for a density $\Lambda = L\mathbf{v}$, $L \in C^\infty(J^1 E)$ on p, are the sections $\bar{s}\colon M \to J^1 E$ of the canonical projection $p_1\colon J^1 E \to M$ such that,

$$\bar{s}^* \left(i_X d\Theta_\Lambda \right) = 0, \quad \forall X \in \mathfrak{X}^v(J^1 E), \tag{6}$$

where $\Theta_\Lambda = (-1)^{i-1} (\partial L / \partial y_i^\alpha) \theta^\alpha \wedge \mathbf{v}_i + L\mathbf{v}$ is the Poincaré-Cartan form attached to Λ (cf. Goldschmidt and Sternberg (1973); Muñoz Masqué and Coronado (2000)), $\mathfrak{X}^v(J^1 E)$ denotes the Lie algebra of p_1-vertical vector fields, $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$ are the standard contact forms on the 1-jet bundle, $(x^i, y^\alpha, y_i^\alpha)_{1 \le i \le n, 1 \le \alpha \le m}$ being the induced coordinate system on $J^1 E$ by a fibred system (x^i, y^α) for the submersion p adapted to the given volume form, i.e., $\mathbf{v} = dx^1 \wedge \ldots \wedge dx^n$, and $\mathbf{v}_i = dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n$.

If $\bar{s} = j^1 s$ is a holonomic section, then $\bar{s}$ is a solution to the Hamilton-Cartan equations if and only if s is a solution to the Euler-Lagrange equations. If Λ is regular, then the converse holds true: Every solution to the Hamilton-Cartan equations is of the form $\bar{s} = j^1 s$, s being a solution to the Euler-Lagrange equations. Hence, for regular variational problems H-C equations are equivalent to E-L equations; but this is no longer true for non-regular densities, as is the case of the Yang-Mills Lagrangian.

As a simple computation shows, a section $\bar{s}$ is a solution to (6) if and only if the following equations hold:

$$\begin{cases} -\dfrac{\partial}{\partial x^j}\left(\dfrac{\partial L}{\partial y_j^\alpha}\circ \bar{s}\right) + \dfrac{\partial L}{\partial y^\alpha}\circ \bar{s} + \left(s_j^\beta - \bar{s}_j^\beta\right)\left(\dfrac{\partial^2 L}{\partial y^\alpha \partial y_j^\beta}\circ \bar{s}\right) = 0, \\ 1 \le \alpha \le m, \end{cases} \tag{7}$$

$$\left(s_j^\beta - \bar{s}_j^\beta\right)\left(\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta}\circ \bar{s}\right) = 0, \quad 1 \le i \le n, 1 \le \alpha \le m, \tag{8}$$

where $s^\alpha = y^\alpha \circ \bar{s}$, $s_i^\alpha = \partial s^\alpha / \partial x^i$, $\bar{s}_i^\alpha = y_i^\alpha \circ \bar{s}$. The Yang-Mills Lagrangian (4) defined on the bundle of connections $E = C$ of the principal bundle P, is written as

$$L = \langle B_\alpha, B_\beta \rangle \Omega_{ij}^\alpha \Omega_{kl}^\beta \Delta^{ij,kl} \det(g_{ab})^{\frac{1}{2}}, \quad i < j, k < l, \tag{9}$$

where $(B_1, \dots, B_m)$ is a basis for $\mathfrak{g}$, $\left[B_\beta, B_\gamma\right] = c_{\beta\gamma}^\alpha B_\alpha$, and

$$\Omega_{ij}^\alpha = A_{i,j}^\alpha - A_{j,i}^\alpha - c_{\beta\gamma}^\alpha A_i^\beta A_j^\gamma, \quad \Delta^{ij,kl} = g^{ik} g^{jl} - g^{il} g^{jk},$$
$$(g^{ij}) = (g_{ij})^{-1}, \quad g = g_{ab} dx^a \otimes dx^b,$$

$(x^i, A_j^\alpha; A_{j,k}^\alpha)$ being the coordinate system on $J^1 C$ induced by a natural coordinate system (x^i, A_j^α) on C; for the details we refer the reader to Castrillón López and Muñoz Masqué (2001).

We now study (7) and (8) for the Lagrangian (9). We assume the basis (B_α) to be orthogonal with respect to the non-degenerate pairing on $\mathfrak{g}$. The Eq. (8) becomes,

$$\left(A_{l,j}^\beta - \bar{A}_{l,j}^\beta\right)\langle B_\alpha, B_\beta \rangle \Delta^{kilj} \det(g_{ab})^{\frac{1}{2}} = 0, \quad \forall \alpha, k, i. \tag{10}$$

As $\det(g_{ab}) \neq 0$, by taking $\alpha = \beta$ and noting that the factor $\Delta^{ki,lj}$ is skew-symmetric with respect to the indices lj, we conclude that $A_{l,j}^\beta - \bar{A}_{l,j}^\beta$ is symmetric. Since $p_{10}\colon J^1 C \to C$ is an affine bundle modelled over $\otimes^2 T^* M \otimes \operatorname{ad} P$, this condition of symmetry can geometrically be expressed by saying that the difference

$j^1 s - \bar{s}$ belongs to $\Gamma(S^2 T^* M \otimes \operatorname{ad} P)$. Moreover, the Eq. (7) reads

$$-\frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial A^\alpha_{i,j}} \circ \bar{s}\right) + \frac{\partial L}{\partial A^\alpha_i} \circ \bar{s} + \left(A^\beta_{k,j} - \bar{A}^\beta_{k,j}\right)\left(\frac{\partial^2 L}{\partial A^\alpha_i \partial A^\beta_{k,j}} \circ \bar{s}\right) = 0,$$

for all α, i. The term

$$\left(A^\beta_{k,j} - \bar{A}^\beta_{k,j}\right)\frac{\partial^2 L}{\partial A^\alpha_i \partial A^\beta_{k,j}} = 4\langle B_\gamma, B_\beta\rangle\left(A^\beta_{k,j} - \bar{A}^\beta_{k,j}\right)c^\gamma_{\alpha\tau} A^\tau_l \Omega^\alpha_{ij} \Delta^{il,kj} \det(g_{ab})^{\frac{1}{2}},$$

identically vanishes, due to the condition (10) and then, the equations above become

$$-\frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial A^\alpha_{i,j}} \circ \bar{s}\right) + \frac{\partial L}{\partial A^\alpha_i} \circ \bar{s} = 0,$$

which are the Euler-Lagrange equations of L, but considering the curvature form $\Omega^{\bar{s}} = \Omega \circ \bar{s}$ instead of the curvature $\Omega^\Gamma = \Omega \circ j^1 s$ of the connection Γ defined by s, where $\Omega\colon J^1 C \to \wedge^2 T^* M \otimes \operatorname{ad} P$ is the curvature mapping. These equations are $\nabla^\Gamma \star \Omega^{\bar{s}} = 0$. We easily check that $\Omega^{\bar{s}} = \Omega^\Gamma$, as $j^1 s - \bar{s}$ is a symmetric 2-tensor. In summary,

Theorem 3.2. *The Hamilton-Cartan equations of the Yang-Mills Lagrangian* 9 *for a section* $\bar{s}\colon M \to J^1 C$ *of* $p_1\colon J^1 C \to M$ *are:* 1) *the standard Yang-Mills equation* $\nabla^\Gamma \star \Omega^\Gamma = 0$, *and* 2) *the condition on* $j^1 s - \bar{s}$ *of being symmetric, where* $s\colon M \to C$ *is* $s = p_{10} \circ \bar{s}$ *and* Γ *denotes the principal connection tautologically defined by* s*. Consequently, both equations are independent of the pairing chosen.*

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